Problemas sugeridos de Cadenas de Markov

(Tomados del Hillier y Lieberman)

16.2-1. Assume that the probability of rain tomorrow is 0.5 if it is raining today, and assume that the probability of its being clear (no rain) tomorrow is 0.9 if it is clear today. Also assume that these probabilities do not change if information is also provided about the weather before today.

(a) Explain why the stated assumptions imply that the *Markovian property* holds for the evolution of the weather.

(b) Formulate the evolution of the weather as a Markov chain by defining its states and giving its (one-step) transition matrix.

(c) Determine the steady-state probabilities.

16.3-3.* A particle moves on a circle through points that have been marked 0, 1, 2, 3, 4 (in a clockwise order). The particle starts at point 0. At each step it has probability 0.5 of moving one point clockwise (0 follows 4) and 0.5 of moving one point counterclockwise. Let X_n $(n \geq 0)$ denote its location on the circle after step n. $\{X_n\}$ is a Markov chain.

(a) Construct the (one-step) transition matrix.

16.4-2. Given each of the following (one-step) transition matrices of a Markov chain, determine the classes of the Markov chain and whether they are recurrent.

$$\text{(a) } \mathbf{P} = \begin{array}{c} \text{State} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \end{bmatrix} \end{array}$$

$$\text{(b) } \mathbf{P} = \begin{array}{c} \text{State} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{bmatrix} 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 1 & 0 \end{bmatrix} \end{array}$$

16.4-3. Given the following (one-step) transition matrix of a Markov chain, determine the classes of the Markov chain and whether they are recurrent.

$$\mathbf{P} = \begin{array}{c} \text{State} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} \frac{1}{4} & \frac{3}{4} & 0 & 0 & 0 \\ \frac{3}{4} & \frac{1}{4} & 0 & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 \\ 0 & 0 & 0 & \frac{3}{4} & \frac{1}{4} \\ 0 & 0 & 0 & \frac{1}{4} & \frac{3}{4} \end{bmatrix} \end{array}$$

16.5-2. A transition matrix **P** is said to be doubly stochastic if the sum over each column equals 1; that is,

$$\sum_{i=0}^{M} p_{ij} = 1, \qquad \text{for all } j.$$

If such a chain is irreducible, aperiodic, and consists of $M + 1$ states, show that

$$\pi_j = \frac{1}{M+1}, \qquad \text{for } j = 0, 1, \ldots, M.$$

16.5-3. Reconsider Prob. 16.3-3. Use the results given in Prob. 16.5-2 to find the steady-state probabilities for this Markov chain. Then find what happens to these steady-state probabilities if, at each step, the probability of moving one point clockwise changes to 0.9 and the probability of moving one point counterclockwise changes to 0.1.

C **16.5-4.** The leading brewery on the West Coast (labeled *A*) has hired an OR analyst to analyze its market position. It is particularly concerned about its major competitor (labeled *B*). The analyst believes that brand switching can be modeled as a Markov chain using three states, with states *A* and *B* representing customers drinking beer produced from the aforementioned breweries and state *C* representing all other brands. Data are taken monthly, and the analyst has constructed the following (one-step) transition matrix from past data.

	A	*B*	*C*
A	0.7	0.2	0.1
B	0.2	0.75	0.05
C	0.1	0.1	0.8

What are the steady-state market shares for the two major breweries?

16.5-6. A soap company specializes in a luxury type of bath soap. The sales of this soap fluctuate between two levels—"Low" and "High"—depending upon two factors: (1) whether they advertise, and (2) the advertising and marketing of new products being done by competitors. The second factor is out of the company's control, but it is trying to determine what its own advertising policy should be. For example, the marketing manager's proposal is to advertise when sales are low but not to advertise when sales are high. Advertising in any quarter of a year has its primary impact on sales in the *following* quarter. Therefore, at the beginning of each quarter, the needed information is available to forecast accurately whether sales will be low or high that quarter and to decide whether to advertise that quarter.

The cost of advertising is \$1 million for each quarter of a year in which it is done. When advertising is done during a quarter, the probability of having high sales the next quarter is $\frac{1}{2}$ or $\frac{3}{4}$, depending upon whether the current quarter's sales are low or high. These probabilities go down to $\frac{1}{4}$ or $\frac{1}{2}$ when advertising is not done during the current quarter. The company's quarterly profits (excluding advertising costs) are \$4 million when sales are high but only \$2 million when sales are low. (Hereafter, use units of millions of dollars.)

(a) Construct the (one-step) transition matrix for each of the following advertising strategies: (i) never advertise, (ii) always advertise, (iii) follow the marketing manager's proposal.

(b) Determine the steady-state probabilities manually for each of the three cases in part (*a*).

(c) Find the long-run expected average profit (including a deduction for advertising costs) per quarter for each of the three advertising strategies in part (*a*). Which of these strategies is best according to this measure of performance?

16.5-9. Consider the following inventory policy for a certain product. If the demand during a period exceeds the number of items available, this unsatisfied demand is backlogged; i.e., it is filled when the next order is received. Let Z_n $(n = 0, 1, \ldots)$ denote the amount of inventory on hand minus the number of units backlogged before ordering at the end of period n $(Z_0 = 0)$. If Z_n is zero or positive, no orders are backlogged. If Z_n is negative, then $-Z_n$ represents the number of backlogged units and no inventory is on hand. At the end of period n, if $Z_n < 1$, an order is placed for $2m$ units, where m is the smallest integer such that $Z_n + 2m \geq 1$. Orders are filled immediately.

Let $D_1, D_2, \ldots,$ be the demand for a product in periods 1, 2, . . . , respectively. Assume that the D_n are independent and identically distributed random variables taking on the values, 0, 1, 2, 3, 4, each with probability $\frac{1}{5}$. Let X_n denote the amount of stock on hand *after* ordering at the end of period n (where $X_0 = 2$), so that

$$X_n = \begin{cases} X_{n-1} - D_n + 2m & \text{if } X_{n-1} - D_n < 1 \\ X_{n-1} - D_n & \text{if } X_{n-1} - D_n \geq 1 \end{cases} \quad (n = 1, 2, \ldots),$$

when $\{X_n\}$ $(n = 0, 1, \ldots)$ is a Markov chain. It has only two states, 1 and 2, because the only time that ordering will take place is when $Z_n = 0, -1, -2,$ or -3, in which case 2, 2, 4, and 4 units are ordered, respectively, leaving $X_n = 2, 1, 2, 1$, respectively.

(a) Construct the (one-step) transition matrix.

(b) Use the steady-state equations to solve manually for the steady-state probabilities.

(c) Now use the result given in Prob. 16.5-2 to find the steady-state probabilities.

(d) Suppose that the ordering cost is given by $(2 + 2m)$ if an order is placed and zero otherwise. The holding cost per period is Z_n if $Z_n \geq 0$ and zero otherwise. The shortage cost per period is $-4Z_n$ if $Z_n < 0$ and zero otherwise. Find the (long-run) expected average cost per unit time.